

# NONCOMMUTATIVE INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS OF MAXIMAL FUNCTIONS AND APPLICATIONS

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**ABSTRACT.** In this paper, we establish a Marcinkiewicz type interpolation theorem for convex functions of maximal functions in the noncommutative setting. As applications, we prove the noncommutative analogue of the Doob inequality for convex functions of maximal functions on martingales, the analogue of the classical Dunford-Schwartz maximal ergodic inequality for convex functions of positive contractions, and that of Stein's maximal inequality for convex functions of symmetric positive contractions. As a consequence, we obtain the moment Burkholder-Davis-Gundy inequality for noncommutative martingales.

## 1. INTRODUCTION

Given a probability space  $(\Omega, \mathcal{F}, P)$ , let  $\{\mathcal{F}_n\}_{n \geq 1}$  be a nondecreasing sequence of  $\sigma$ -subfields of  $\mathcal{F}$  such that  $\mathcal{F} = \vee \mathcal{F}_n$ . Suppose that  $\Phi$  is a convex function from  $[0, \infty)$  into  $[0, \infty)$  satisfying  $\Phi(0) = 0$  and the growth condition  $\Phi(2t) \leq C\Phi(t)$  for any  $t > 0$  (i.e., the so-called  $\Delta_2$ -condition). It was proved by Burkholder, Davis, and Gundy in [9] that

$$(1.1) \quad \int_{\Omega} \Phi \left[ \left( \sum_{n=1}^{\infty} |df_n|^2 \right)^{\frac{1}{2}} \right] dP \approx \int_{\Omega} \Phi \left( \sup_{n \geq 1} |f_n| \right) dP,$$

for all martingales  $f = (f_n)_{n \geq 1}$  based on  $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \geq 1}, P)$ , where  $df = (df_n)_{n \geq 1}$  is the martingale difference of  $f$ . For convex powers  $\Phi(t) = t^p$ , this result was proved by Burkholder [7] in the case where  $1 < p < \infty$  and for  $p = 1$  by Davis [12]. Also, the inequality holds for  $0 < p < 1$  if  $f$  is restricted, see [10] for detailed information. As shown in [9] (see also [8]), (1.1) has important implications for continuous-time martingales.

However, the noncommutative case is surprisingly different. Indeed, it was shown in [20], Corollary 14, that (1.1) does not hold for  $\Phi(t) = t$  in general. A natural question arises: what assumptions on  $\Phi$  imply the validity of (1.1) in the noncommutative setting? Our answer to this question, which is one consequence of our main results in this paper, reads as follows: (See Section 2 for the definitions of  $p_{\Phi}$  and  $q_{\Phi}$ .)

**Theorem 1.1.** *Let  $\mathcal{M}$  be a finite von Neumann algebra with a normalized normal faithful trace  $\tau$ , equipped with a filtration  $(\mathcal{M}_n)_{n \geq 0}$  of von Neumann subalgebras of  $\mathcal{M}$ . Let  $\Phi$  be an Orlicz function, and let  $x = (x_n)_{n \geq 0}$  be a noncommutative  $L_{\Phi}$ -martingale with respect to  $(\mathcal{M}_n)_{n \geq 0}$ . If  $1 < p_{\Phi} \leq q_{\Phi} < 2$ , then*

$$(1.2) \quad \begin{aligned} & \tau \left( \Phi \left[ \sup_n^+ x_n \right] \right) \\ & \approx \inf \left\{ \tau \left( \Phi \left[ \left( \sum_{k=0}^{\infty} |dy_k|^2 \right)^{\frac{1}{2}} \right] \right) + \tau \left( \Phi \left[ \left( \sum_{k=0}^{\infty} |dz_k^*|^2 \right)^{\frac{1}{2}} \right] \right) \right\}, \end{aligned}$$

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where the infimum runs over all decomposition  $x_n = y_n + z_n$  with  $y_n$  in  $\mathcal{H}_\Phi^C(\mathcal{M})$  and  $z_n$  in  $\mathcal{H}_\Phi^R(\mathcal{M})$ ; and if  $2 < p_\Phi \leq q_\Phi < \infty$ , then

$$(1.3) \quad \begin{aligned} & \tau\left(\Phi\left[\sup_n^+ x_n\right]\right) \\ & \approx \max\left\{\tau\left(\Phi\left[\left(\sum_{k=0}^{\infty} |dx_k|^2\right)^{\frac{1}{2}}\right]\right), \tau\left(\Phi\left[\left(\sum_{k=0}^{\infty} |dx_k^*|^2\right)^{\frac{1}{2}}\right]\right)\right\}. \end{aligned}$$

Here, the definition of  $\tau(\Phi[\sup_n^+ x_n])$  for a noncommutative sequence  $(x_n)$  will be directly derived from Pisier's theory of vector-valued noncommutative  $L_p$ -spaces. In fact, we will present a slight variant of Pisier's definition on maximal functions in the noncommutative setting (see Sect. 3, Definition 3.2 below).

Note that stopping times and good- $\lambda$  techniques developed by Burkholder et al. (see [8] for details) are two key ingredients in the proof of (1.1). However, these concepts do not appear to have a trackable non-commutative extension (there are some works on this topic, see [2] and references therein). These deficiencies have been overcome recently in the theory of noncommutative martingales. In fact, most of the classical martingale inequalities have been successfully transferred to the noncommutative setting. This general theme started from the fundamental paper of Pisier and Xu [30] where they formulated the right analogue of Burkholder-Gundy inequalities. It was their general functional analytic approach that led to the renewed interests in this topic, see for example [6, 18, 19, 20, 22, 27, 28, 32, 33, 34] and references therein. We also refer the reader to a recent book by Xu [36] for a comprehensive exposition of the subject.

In particular, using the techniques developed for noncommutative martingales, as well as operator space theory and theory of interpolation of Banach spaces, Junge and Xu [21] proved a noncommutative analogue of the classical Dunford-Schwartz maximal ergodic inequality for positive contraction on  $L_p$ , and the analogue of Stein's maximal inequality for symmetric positive contractions on  $L_p$ ,  $1 < p < \infty$ . As a consequence, they obtained a generalization of Yeadon's theorem [37] to noncommutative  $L_p$  for all  $1 < p < \infty$ . The maximal inequalities they obtained constitute a significant breakthrough in the noncommutative ergodic theory. Their result on Stein's maximal inequality was further extended by the first named author [3].

We will establish Theorem 1.1 by proving the following noncommutative extension of Doob's martingale inequality.

**Theorem 1.2.** *Let  $\mathcal{M}$  be a finite von Neumann algebra with a normalized normal faithful trace  $\tau$ , equipped with a filtration  $(\mathcal{M}_n)_{n \geq 0}$  of von Neumann subalgebras of  $\mathcal{M}$ . Let  $\Phi$  be an Orlicz function and  $x = (x_n)$  be a noncommutative  $L_\Phi$ -martingale with respect to  $(\mathcal{M}_n)$ . If  $1 < p_\Phi \leq q_\Phi < \infty$ , then*

$$(1.4) \quad \tau\left[\Phi\left(\sup_n^+ x_n\right)\right] \approx \tau[\Phi(|x|)].$$

Our key ingredient is a noncommutative extension of Marcinkiewicz type interpolation theorem for convex functions of maximal functions, which we will prove in this paper. We note that the formulation of this was directly derived from Pisier's theory of vector-valued noncommutative  $L_p$ -spaces (see [18, 29]). Our proofs combine arguments developed by Junge and Xu in [21] and by the present authors in [4], respectively.

This interpolation result is also applied to obtain the noncommutative analogue of the classical maximal ergodic inequality for convex functions of contractions. To state this, we need some notation.

Let  $\mathcal{M}$  be a semifinite von Neumann algebra with a normal semifinite faithful trace  $\tau$ , and let  $L_p(\mathcal{M})$  be the associated noncommutative  $L_p$ -space. Consider a linear map  $T : \mathcal{M} \mapsto \mathcal{M}$  which may satisfy the following conditions:

- (I)  $T$  is a contraction on  $\mathcal{M}$ , that is,  $\|Tx\| \leq \|x\|$  for all  $x \in \mathcal{M}$ .

- (II)  $T$  is positive, i.e.,  $Tx \geq 0$  if  $x \geq 0$ .
  - (III)  $\tau \circ T \leq \tau$ , that is,  $\tau(Tx) \leq \tau(x)$  for all  $x \in L_1(\mathcal{M}) \cap \mathcal{M}_+$ .
  - (IV)  $T$  is symmetric relative to  $\tau$ , i.e.,  $\tau((Ty)^*x) = \tau(y^*Tx)$  for all  $x, y \in L_2(\mathcal{M}) \cap \mathcal{M}$ .
- Under conditions (I)-(III),  $T$  naturally extends to a contraction on  $L_p(\mathcal{M})$  for every  $1 \leq p < \infty$ . The extension will be still denoted by  $T$ .

The following is the second main result of this paper.

**Theorem 1.3.** *Let  $\Phi$  be an Orlicz function with  $1 < p_\Phi \leq q_\Phi < \infty$ . If  $T : \mathcal{M} \mapsto \mathcal{M}$  is a linear map satisfying (I) – (III), then*

$$(1.5) \quad \tau \left( \Phi \left[ \sup_n^+ M_n(x) \right] \right) \lesssim \tau(\Phi[|x|]), \quad \forall x \in L_\Phi(\mathcal{M}),$$

where  $M_n := \frac{1}{n+1} \sum_{k=0}^n T^k$  for any  $n \geq 1$ . If, in addition,  $T$  satisfies (IV), then

$$(1.6) \quad \tau \left( \Phi \left[ \sup_n^+ T^n(x) \right] \right) \lesssim \tau(\Phi[|x|]), \quad \forall x \in L_\Phi(\mathcal{M}).$$

The inequality (1.5) is the noncommutative analogue of the classical Dunford-Schwartz maximal ergodic inequality for convex functions of positive contractions, while (1.6) is the analogue of Stein's maximal inequality for convex functions of symmetric positive contractions. As mentioned previously, the proofs of (1.5) and (1.6) are again based on our noncommutative integral inequalities for convex functions of maximal functions.

The paper is organized as follows. In Section 2, we present some preliminaries and notations on noncommutative Orlicz spaces, Orlicz-Hardy spaces of noncommutative martingales, and maximal functions in the noncommutative setting. Then, a noncommutative analogue of Marcinkiewicz type interpolation theorem for convex functions of maximal functions is proved in Section 3, which is the key ingredient of the proofs of our results. In Section 4 we establish Theorems 1.1, 1.2, and 1.3. Finally, in Section 5, the results obtained in the previous sections are extended to cover weak type inequalities, as well as maximal inequalities on noncommutative symmetric spaces.

In what follows,  $C$  always denotes a constant, which may be different in different places. For two nonnegative (possibly infinite) quantities  $X$  and  $Y$  by  $X \lesssim Y$  we mean that there exists a constant  $C > 0$  such that  $X \leq CY$ , and by  $X \approx Y$  that  $X \lesssim Y$  and  $Y \lesssim X$ .

## 2. PRELIMINARIES

**2.1. Noncommutative Orlicz spaces.** We use standard notions from theory of noncommutative  $L_p$ -spaces. Our main references are [31] and [36] (see also [31] for more historical references). Let  $\mathcal{N}$  be a semifinite von Neumann algebra acting on a Hilbert space  $\mathbb{H}$  with a normal semifinite faithful trace  $\nu$ . Let  $L_0(\mathcal{N})$  denote the topological  $*$ -algebra of measurable operators with respect to  $(\mathcal{N}, \nu)$ . The topology of  $L_0(\mathcal{N})$  is determined by the convergence in measure. The trace  $\nu$  can be extended to the positive cone  $L_0^+(\mathcal{N})$  of  $L_0(\mathcal{N})$ :

$$\nu(x) = \int_0^\infty \lambda d\nu(E_\lambda(x)),$$

where  $x = \int_0^\infty \lambda dE_\lambda(x)$  is the spectral decomposition of  $x$ . Given  $0 < p < \infty$ , let

$$L_p(\mathcal{N}) = \{x \in L_0(\mathcal{N}) : \nu(|x|^p)^{\frac{1}{p}} < \infty\}.$$

We define

$$\|x\|_p = \nu(|x|^p)^{\frac{1}{p}}, \quad x \in L_p(\mathcal{N}).$$

Then  $(L_p(\mathcal{N}), \|\cdot\|_p)$  is a Banach (or quasi-Banach for  $p < 1$ ) space. This is the noncommutative  $L_p$ -space associated with  $(\mathcal{N}, \nu)$ , denoted by  $L_p(\mathcal{N}, \nu)$  or simply by  $L_p(\mathcal{N})$ . As usual, we set  $L_\infty(\mathcal{N}, \nu) = \mathcal{N}$  equipped with the operator norm.

For  $x \in L_0(\mathcal{N})$  we define

$$\lambda_s(x) = \tau(e_s^\perp(|x|)) \quad (s > 0) \quad \text{and} \quad \mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leq t\} \quad (t > 0),$$

where  $e_s^\perp(|x|) = e_{(s,\infty)}(|x|)$  is the spectral projection of  $|x|$  associated with the interval  $(s, \infty)$ . The function  $s \mapsto \lambda_s(x)$  is called the *distribution function* of  $x$  and  $\mu_t(x)$  is the *generalized singular number* of  $x$ . We will denote simply by  $\lambda(x)$  and  $\mu(x)$  the functions  $s \mapsto \lambda_s(x)$  and  $t \mapsto \mu_t(x)$ , respectively. It is easy to check that both are decreasing and continuous from the right on  $(0, \infty)$ . For further information we refer the reader to [17].

For  $0 < p < \infty$ , we have the Kolmogorov inequality

$$(2.1) \quad \lambda_s(x) \leq \frac{\|x\|_p^p}{s^p}, \quad \forall s > 0,$$

for any  $x \in L_p(\mathcal{N})$ . If  $x, y$  in  $L_0(\mathcal{N})$ , then

$$(2.2) \quad \lambda_{2s}(x+y) \leq \lambda_s(x) + \lambda_s(y), \quad \forall s > 0.$$

Let  $\Phi$  be an Orlicz function on  $[0, \infty)$ , i.e., a continuous increasing and convex function satisfying  $\Phi(0) = 0$  and  $\lim_{t \rightarrow \infty} \Phi(t) = \infty$ . Recall that  $\Phi$  is said to satisfy the  $\Delta_2$ -condition if there is a constant  $C$  such that  $\Phi(2t) \leq C\Phi(t)$  for all  $t > 0$ . In this case, we write  $\Phi \in \Delta_2$ . It is easy to check that  $\Phi \in \Delta_2$  if and only if for any  $a > 0$  there is a constant  $C_a > 0$  such that  $\Phi(at) \leq C_a\Phi(t)$  for all  $t > 0$ .

For any  $x \in L_0(\mathcal{N})$ , by means of functional calculus applied to the spectral decomposition of  $|x|$ , we have

$$\nu(\Phi(|x|)) = \int_0^\infty \lambda_s(|x|) d\Phi(s) = \int_0^\infty \Phi(\mu_t(x)) dt,$$

(e.g., see [17].) Recall that for any  $x, y \in L_0(\mathcal{N})$  there exist two partial isometries  $u, v \in \mathcal{N}$  such that

$$(2.3) \quad |x+y| \leq u^*|x|u + v^*|y|v,$$

(see [1]). Then, we have

$$\nu(\Phi(|\alpha x + (1-\alpha)y|)) \leq \alpha\nu(\Phi(|x|)) + (1-\alpha)\nu(\Phi(|y|))$$

for any  $0 \leq \alpha \leq 1$  and  $x, y \in L_0(\mathcal{N})$ . In addition, if  $\Phi \in \Delta_2$ , then

$$\nu(\Phi(|x+y|)) \leq C_\Phi [\nu(\Phi(|x|)) + \nu(\Phi(|y|))].$$

We will frequently use these two inequalities in what follows.

We will work with some standard indices associated to an Orlicz function. Given an Orlicz function  $\Phi$ , let

$$M(t, \Phi) = \sup_{s>0} \frac{\Phi(ts)}{\Phi(s)}, \quad t > 0.$$

Define

$$p_\Phi = \lim_{t \searrow 0} \frac{\log M(t, \Phi)}{\log t}, \quad q_\Phi = \lim_{t \nearrow \infty} \frac{\log M(t, \Phi)}{\log t}.$$

Note the following properties:

- (1)  $1 \leq p_\Phi \leq q_\Phi \leq \infty$ .
- (2) The following characterizations of  $p_\Phi$  and  $q_\Phi$  hold

$$p_\Phi = \sup \left\{ p > 0 : \int_0^t s^{-p} \Phi(s) \frac{ds}{s} = O(t^{-p} \Phi(t)), \forall t > 0 \right\};$$

$$q_\Phi = \inf \left\{ q > 0 : \int_t^\infty s^{-q} \Phi(s) \frac{ds}{s} = O(t^{-q} \Phi(t)), \forall t > 0 \right\}.$$

- (3)  $\Phi \in \Delta_2$  if and only if  $q_\Phi < \infty$ , or equivalently,  $\sup_{t>0} t\Phi'(t)/\Phi(t) < \infty$ . ( $\Phi'(t)$  is defined for each  $t > 0$  except for a countable set of points in which we take  $\Phi'(t)$  as the derivative from the right.)

See [25, 26] for more information on Orlicz functions and Orlicz spaces.

For an Orlicz function  $\Phi$ , the noncommutative Orlicz space  $L_\Phi(\mathcal{N})$  is defined as the space of all measurable operators  $x$  with respect to  $(\mathcal{N}, \nu)$  such that

$$\nu\left(\Phi\left(\frac{|x|}{c}\right)\right) < \infty$$

for some  $c > 0$ . The space  $L_\Phi(\mathcal{N})$ , equipped with the norm

$$\|x\|_\Phi = \inf \{c > 0 : \nu(\Phi(|x|/c)) < 1\},$$

is a Banach space. If  $\Phi(t) = t^p$  with  $1 \leq p < \infty$  then  $L_\Phi(\mathcal{N}) = L_p(\mathcal{N})$ . Note that if  $\Phi \in \Delta_2$ , then for  $x \in L_0(\mathcal{N})$ ,  $\nu(\Phi(x)) < \infty$  if and only if  $x \in L_\Phi(\mathcal{N})$ . Noncommutative Orlicz spaces are symmetric spaces of measurable operators as defined in [15, 35].

Let  $a = (a_n)$  be a finite sequence in  $L_\Phi(\mathcal{N})$ . We define

$$\|a\|_{L_\Phi(\mathcal{N}, \ell_C^2)} = \left\| \left( \sum_n |a_n|^2 \right)^{\frac{1}{2}} \right\|_\Phi \quad \text{and} \quad \|a\|_{L_\Phi(\mathcal{N}, \ell_R^2)} = \left\| \left( \sum_{n \geq 0} |a_n^*|^2 \right)^{\frac{1}{2}} \right\|_\Phi,$$

respectively. This gives two norms on the family of all finite sequences in  $L_\Phi(\mathcal{N})$  (see [4] for details). The corresponding completion  $L_\Phi(\mathcal{N}, \ell_C^2)$  is a Banach space. It is clear that a sequence  $a = (a_n)_{n \geq 0}$  in  $L_\Phi(\mathcal{N})$  belongs to  $L_\Phi(\mathcal{N}, \ell_C^2)$  if and only if

$$\sup_{n \geq 0} \left\| \left( \sum_{k=0}^n |a_k|^2 \right)^{\frac{1}{2}} \right\|_\Phi < \infty.$$

If this is the case,  $\left( \sum_{k=0}^\infty |a_k|^2 \right)^{\frac{1}{2}}$  can be appropriately defined as an element of  $L_\Phi(\mathcal{N})$ . Similarly,  $\|\cdot\|_{L_\Phi(\mathcal{N}, \ell_R^2)}$  is also a norm on the family of all finite sequence in  $L_\Phi(\mathcal{N})$ , and the corresponding completion  $L_\Phi(\mathcal{N}, \ell_R^2)$  is a Banach space, which is isometric to the row subspace of  $L_\Phi(\mathcal{N} \otimes \mathcal{B}(\ell^2))$  consisting of matrices whose nonzero entries lie only in the first row. Observe that the column and row subspaces of  $L_\Phi(\mathcal{N} \otimes \mathcal{B}(\ell^2))$  are 1-complemented by Theorem 3.4 in [16].

In what follows, unless otherwise specified, we always denote by  $\Phi$  an Orlicz function.

**2.2. Noncommutative martingales.** Let  $\mathcal{M}$  be a finite von Neumann algebra with a normalized normal faithful trace  $\tau$ . Let  $(\mathcal{M}_n)_{n \geq 0}$  be an increasing sequence of von Neumann subalgebras of  $\mathcal{M}$  such that  $\cup_{n \geq 0} \mathcal{M}_n$  generates  $\mathcal{M}$  (in the  $w^*$ -topology).  $(\mathcal{M}_n)_{n \geq 0}$  is called a filtration of  $\mathcal{M}$ . The restriction of  $\tau$  to  $\mathcal{M}_n$  is still denoted by  $\tau$ . Let  $\mathcal{E}_n = \mathcal{E}(\cdot | \mathcal{M}_n)$  be the conditional expectation of  $\mathcal{M}$  with respect to  $\mathcal{M}_n$ . Then  $\mathcal{E}_n$  is a norm 1 projection of  $L_\Phi(\mathcal{M})$  onto  $L_\Phi(\mathcal{M}_n)$  (see Theorem 3.4 in [16]) and  $\mathcal{E}_n(x) \geq 0$  whenever  $x \geq 0$ .

A noncommutative  $L_\Phi$ -martingale with respect to  $(\mathcal{M}_n)_{n \geq 0}$  is a sequence  $x = (x_n)_{n \geq 0}$  such that  $x_n \in L_\Phi(\mathcal{M}_n)$  and

$$\mathcal{E}_n(x_{n+1}) = x_n$$

for any  $n \geq 0$ . Let  $\|x\|_\Phi = \sup_{n \geq 0} \|x_n\|_\Phi$ . If  $\|x\|_\Phi < \infty$ , then  $x$  is said to be a bounded  $L_\Phi$ -martingale.

**Remark 2.1.** Let  $\mathcal{M}$  be a semifinite von Neumann algebra with a semifinite normal faithful trace  $\tau$ . Let  $(\mathcal{M}_n)_{n \geq 0}$  be a filtration of  $\mathcal{M}$  such that the restriction of  $\tau$  to each  $\mathcal{M}_n$  is still semifinite. Then we can define noncommutative martingales with respect to  $(\mathcal{M}_n)_{n \geq 0}$ . All results on noncommutative martingales that will be presented below can be extended to this semifinite setting.

Let  $x$  be a noncommutative martingale. The martingale difference sequence of  $x$ , denoted by  $dx = (dx_n)_{n \geq 0}$ , is defined as

$$dx_0 = x_0, \quad dx_n = x_n - x_{n-1}, \quad n \geq 1.$$

Set

$$S_n^C(x) = \left( \sum_{k=0}^n |dx_k|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad S_n^R(x) = \left( \sum_{k=0}^n |dx_k^*|^2 \right)^{\frac{1}{2}}.$$

By the preceding discussion,  $dx$  belongs to  $L_\Phi(\mathcal{M}, \ell_C^2)$  (resp.  $L_\Phi(\mathcal{M}, \ell_R^2)$ ) if and only if  $(S_n^C(x))_{n \geq 0}$  (resp.  $(S_n^R(x))_{n \geq 0}$ ) is a bounded sequence in  $L_\Phi(\mathcal{M})$ ; in this case,

$$S^C(x) = \left( \sum_{k=0}^{\infty} |dx_k|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad S^R(x) = \left( \sum_{k=0}^{\infty} |dx_k^*|^2 \right)^{\frac{1}{2}}$$

are elements in  $L_\Phi(\mathcal{M})$ . These are noncommutative analogues of the usual square functions in the commutative martingale theory. It should be pointed out that the two sequences  $S_n^C(x)$  and  $S_n^R(x)$  may not be bounded in  $L_\Phi(\mathcal{M})$  at the same time.

We define  $\mathcal{H}_\Phi^C(\mathcal{M})$  (resp.  $\mathcal{H}_\Phi^R(\mathcal{M})$ ) to be the space of all  $L_\Phi$ -martingales such that  $dx \in L_\Phi(\mathcal{M}, \ell_C^2)$  (resp.  $dx \in L_\Phi(\mathcal{M}, \ell_R^2)$ ), equipped with the norm

$$\|x\|_{\mathcal{H}_\Phi^C(\mathcal{M})} = \|dx\|_{L_\Phi(\mathcal{M}, \ell_C^2)} \quad (\text{resp. } \|x\|_{\mathcal{H}_\Phi^R(\mathcal{M})} = \|dx\|_{L_\Phi(\mathcal{M}, \ell_R^2)}).$$

$\mathcal{H}_\Phi^C(\mathcal{M})$  and  $\mathcal{H}_\Phi^R(\mathcal{M})$  are Banach spaces. Note that if  $x \in \mathcal{H}_\Phi^C(\mathcal{M})$ ,

$$\|x\|_{\mathcal{H}_\Phi^C(\mathcal{M})} = \sup_{n \geq 0} \|S_n^C(x)\|_{L_\Phi(\mathcal{M})} = \|S^C(x)\|_{L_\Phi(\mathcal{M})}.$$

Similar equalities hold for  $\mathcal{H}_\Phi^R(\mathcal{M})$ .

Now, we define the Orlicz-Hardy spaces of noncommutative martingales as follows: If  $p_\Phi < 2$ , then

$$\mathcal{H}_\Phi(\mathcal{M}) = \mathcal{H}_\Phi^C(\mathcal{M}) + \mathcal{H}_\Phi^R(\mathcal{M}),$$

equipped with the norm

$$\|x\| = \inf \left\{ \|y\|_{\mathcal{H}_\Phi^C(\mathcal{M})} + \|z\|_{\mathcal{H}_\Phi^R(\mathcal{M})} : x = y + z, y \in \mathcal{H}_\Phi^C(\mathcal{M}), z \in \mathcal{H}_\Phi^R(\mathcal{M}) \right\}.$$

If  $2 \leq p_\Phi$ ,

$$\mathcal{H}_\Phi(\mathcal{M}) = \mathcal{H}_\Phi^C(\mathcal{M}) \cap \mathcal{H}_\Phi^R(\mathcal{M}),$$

equipped with the norm

$$\|x\| = \max \left\{ \|x\|_{\mathcal{H}_\Phi^C(\mathcal{M})}, \|x\|_{\mathcal{H}_\Phi^R(\mathcal{M})} \right\}.$$

We refer to [4] for more information on  $\mathcal{H}_\Phi(\mathcal{M})$ .

**2.3. The space  $L_p(\mathcal{M}; \ell^\infty)$ .** Given  $1 \leq p < \infty$ , recall that  $L_p(\mathcal{M}; \ell^\infty)$  is defined as the space of all sequences  $(x_n)_{n \geq 1}$  in  $L_p(\mathcal{M})$  for which there exist  $a, b \in L_{2p}(\mathcal{M})$  and a bounded sequence  $(y_n)_{n \geq 1}$  in  $\mathcal{M}$  such that  $x_n = ay_nb$  for all  $n \geq 1$ . For such a sequence, set

$$(2.4) \quad \|(x_n)_{n \geq 1}\|_{L_p(\mathcal{M}, \ell^\infty)} := \inf \left\{ \|a\|_{2p} \sup_n \|y_n\|_\infty \|b\|_{2p} \right\},$$

where the infimum runs over all possible factorizations of  $(x_n)_{n \geq 1}$  as above. This is a norm and  $L_p(\mathcal{M}; \ell^\infty)$  is a Banach space. These spaces were first introduced by Pisier [29] in the case when  $\mathcal{M}$  is hyperfinite and by Junge [18] in the general case. It is easy to check that

$$(2.5) \quad \|(x_n)_{n \geq 1}\|_{L_p(\mathcal{M}, \ell^\infty)} = \inf \left\{ \frac{1}{2} \left( \|a\|_{2p}^2 + \|b\|_{2p}^2 \right) \sup_n \|y_n\|_\infty \right\},$$

the infimum taken over the same parameters as above.

As in [21], we usually write

$$\left\| \sup_n x_n \right\|_p = \|(x_n)_{n \geq 1}\|_{L_p(\mathcal{M}, \ell^\infty)}.$$

We warn the reader that this suggestive notation should be treated with care. It is used for possibly nonpositive operators and

$$\left\| \sup_n^+ x_n \right\|_p \neq \left\| \sup_n^+ |x_n| \right\|_p$$

in general. However it has an intuitive description in the positive case, as observed in [21] (p.329 there): A positive sequence  $(x_n)_{n \geq 1}$  of  $L_p(\mathcal{M})$  belongs to  $L_p(\mathcal{M}; \ell^\infty)$  if and only if there exists a positive  $a \in L_p(\mathcal{M})$  such that  $x_n \leq a$  for any  $n \geq 1$  and in this case,

$$(2.6) \quad \left\| \sup_n^+ x_n \right\|_p = \inf \left\{ \|a\|_p : a \in L_p(\mathcal{M}), x_n \leq a, \forall n \geq 1 \right\}.$$

In particular, it was proved in [21] that the spaces  $L_p(\mathcal{M}; \ell^\infty)$  for all  $1 \leq p \leq \infty$  form interpolation scales with respect to complex interpolation. However, this result is no longer true for the real interpolation. This is one of the difficulties one will encounter for dealing with Marcinkiewicz type theorem on maximal functions in the noncommutative setting.

### 3. NONCOMMUTATIVE INTEGRAL INEQUALITIES

In this section, we will establish a noncommutative integral inequality for convex functions of maximal functions, which plays a crucial role in the proofs of Theorems 1.1, 1.2, and 1.3.

To this end, we introduce the following definition.

**Definition 3.1.** Let  $1 \leq p_0 < p_1 \leq \infty$ . Let  $S = (S_n)_{n \geq 1}$  be a sequence of maps from  $L_{p_0}^+(\mathcal{M}) + L_{p_1}^+(\mathcal{M}) \mapsto L_0^+(\mathcal{M})$ .

(1)  $S$  is said to be subadditive, if for any  $n \geq 1$ ,

$$S_n(x + y) \leq S_n(x) + S_n(y), \quad \forall x, y \in L_{p_0}^+(\mathcal{M}) + L_{p_1}^+(\mathcal{M}).$$

(2)  $S$  is said to be of weak type  $(p, p)$  ( $p_0 \leq p < p_1$ ) if there is a positive constant  $C$  such that for any  $x \in L_p^+(\mathcal{M})$  and any  $\lambda > 0$  there exists a projection  $e \in \mathcal{M}$  such that

$$\tau(e^\perp) \leq \left( \frac{C \|x\|_p}{\lambda} \right)^p \quad \text{and} \quad e S_n(x) e \leq \lambda, \quad \forall n \geq 1.$$

(3)  $S$  is said to be of type  $(p, p)$  ( $p_0 \leq p \leq p_1$ ) if there is a positive constant  $C$  such that for any  $x \in L_p^+(\mathcal{M})$  there exists  $a \in L_p^+(\mathcal{M})$  satisfying

$$\|a\|_p \leq C \|x\|_p \quad \text{and} \quad S_n(x) \leq a, \quad \forall n \geq 1.$$

In other words,  $S$  is of type  $(p, p)$  if and only if  $\|S(x)\|_{L_p(\mathcal{M}; \ell^\infty)} \leq C \|x\|_p$  for all  $x \in L_p^+(\mathcal{M})$ .

This definition of subadditive operators in the noncommutative setting is due to Junge and Xu [21], who proved a noncommutative analogue of the classical Marcinkiewicz interpolation theorem as follows.

**Theorem 3.1.** ([21], Theorem 3.1) Let  $1 \leq p_0 < p_1 \leq \infty$ . Let  $S = (S_n)_{n \geq 1}$  be a sequence of maps from  $L_{p_0}^+(\mathcal{M}) + L_{p_1}^+(\mathcal{M}) \mapsto L_0^+(\mathcal{M})$ . Assume that  $S$  is subadditive. If  $S$  is of weak type  $(p_0, p_0)$  with constant  $C_0$  and of type  $(p_1, p_1)$  with constant  $C_1$ , then for any  $p_0 < p < p_1$ ,  $S$  is of type  $(p, p)$  with constant  $C_p$  satisfying

$$C_p \leq C C_0^{1-\theta} C_1^\theta \left( \frac{1}{p_0} - \frac{1}{p} \right)^{-2}$$

where  $\theta$  is determined by  $1/p = (1 - \theta)/p_0 + \theta/p_1$  and  $C$  is an absolute constant.

To state our result on noncommutative integral inequalities, we need more notations.

**Definition 3.2.** Let  $\Phi$  be an Orlicz function. Let  $(x_n)$  be a sequence in  $L_\Phi(\mathcal{M})$ . We define  $\tau[\Phi(\sup_n^+ x_n)]$  by

$$(3.1) \quad \tau[\Phi(\sup_n^+ x_n)] := \inf \left\{ \frac{1}{2} \left( \tau[\Phi(|a|^2)] + \tau[\Phi(|b|^2)] \right) \sup_n \|y_n\|_\infty \right\}$$

where the infimum is taken over all decompositions  $x_n = ay_nb$  for  $a, b \in L_0(\mathcal{M})$  and  $(y_n) \subset L_\infty(\mathcal{M})$  with  $|a|^2, |b|^2 \in L_\Phi(\mathcal{M})$ , and  $\|y_n\|_\infty \leq 1$  for all  $n$ .

**Remark 3.1.** This definition is motivated by (2.5), which is a key point of this paper. A direct generalization of the original form (2.4) seems to be invalid for the convex function of maximal functions.

To understand  $\tau[\Phi(\sup_n^+ x_n)]$ , let us consider a positive sequence  $x = (x_n)$  in  $L_\Phi(\mathcal{M})$ . We then note that

$$(3.2) \quad \tau[\Phi(\sup_n^+ x_n)] \leq \tau[\Phi(a)],$$

if  $a \in L_\Phi^+(\mathcal{M})$  such that  $x_n \leq a$  for all  $n$ . Indeed, for every  $n$  there exists a contraction  $u_n$  such that  $x_n^{\frac{1}{2}} = u_n a^{\frac{1}{2}}$  and hence  $x_n = a^{\frac{1}{2}} u_n^* u_n a^{\frac{1}{2}}$ . This concludes (3.2). Moreover, the converse to (3.2) also holds true provided  $\Phi \in \Delta_2$  (see Proposition 3.1 below).

We collect some basic properties of the quantity  $\tau[\Phi(\sup_n^+ x_n)]$ .

**Proposition 3.1.** Let  $\Phi$  be an Orlicz function satisfying the  $\Delta_2$ -condition.

(1) If  $x = (x_n)$  is a positive sequence in  $L_\Phi(\mathcal{M})$ , then

$$\tau[\Phi(\sup_n^+ x_n)] \approx \inf \left\{ \tau[\Phi(a)] : a \in L_\Phi^+(\mathcal{M}) \text{ such that } x_n \leq a, \forall n \geq 1 \right\}.$$

(2) For any two sequences  $x = (x_n), y = (y_n)$  in  $L_\Phi(\mathcal{M})$  one has

$$\tau[\Phi(\sup_n^+(x_n + y_n))] \lesssim \tau[\Phi(\sup_n^+ x_n)] + \tau[\Phi(\sup_n^+ y_n)].$$

*Proof.* (1). Let  $(x_n)$  be a sequence of positive elements in  $L_\Phi(\mathcal{M})$ . Suppose  $x_n = ay_nb$  with  $|a|^2, |b|^2 \in L_\Phi(\mathcal{M})$  and  $\sup_n \|y_n\|_\infty \leq 1$ . Without loss of generality, we can assume  $a, b \geq 0$ . Set  $c = (a^2 + b^2)^{1/2}$ . Then there exist two partial isometries  $u, v \in \mathcal{M}$  such that

$$a = cu \quad \text{and} \quad b = vc,$$

i.e.,  $x_n = c u y_n v c$  for all  $n$ , and  $\sup_n \|u y_n v\|_\infty \leq 1$ . Thus,  $x_k \leq c^2 \sup_n \|y_n\|_\infty$  for all  $k$ . By the  $\Delta_2$ -condition, one has

$$\begin{aligned} \tau[\Phi(c^2 \sup_n \|y_n\|_\infty)] &\leq \sup_n \|y_n\|_\infty \tau[\Phi(c^2)] \\ &\leq C \sup_n \|y_n\|_\infty \frac{1}{2} \left( \tau[\Phi(|a|^2)] + \tau[\Phi(|b|^2)] \right). \end{aligned}$$

Combining this with (3.2) completes the proof of (1).

(2). We have the following useful description of  $\tau[\Phi(\sup_n^+ x_n)]$ :

$$(3.3) \quad \tau[\Phi(\sup_n^+ x_n)] = \inf \left\{ \frac{1}{2} \left( \tau[\Phi(|a|^2)] + \tau[\Phi(|b|^2)] \right) \right\},$$

where the infimum is taken over all decompositions  $x_n = ay_nb$  for  $a, b \in L_0(\mathcal{M})$  and  $(y_n) \subset L_\infty(\mathcal{M})$  with  $|a|^2, |b|^2 \in L_\Phi(\mathcal{M})$ , and  $\sup_n \|y_n\|_\infty = 1$ . Indeed, for a decomposition  $x_n = ay_nb$  with  $\sup_n \|y_n\|_\infty \leq 1$ , we set  $\tilde{a} = \lambda^{1/2} a$ ,  $\tilde{b} = \lambda^{1/2} b$ , and  $\tilde{y}_n = y_n / \lambda$  with  $\lambda = \sup_n \|y_n\|_\infty$ . Then  $x_n = \tilde{a} \tilde{y}_n \tilde{b}$  for all  $n$  and  $\sup_n \|\tilde{y}_n\|_\infty = 1$ , so that

$$\tau[\Phi(|\tilde{a}|^2)] + \tau[\Phi(|\tilde{b}|^2)] \leq \lambda \left( \tau[\Phi(|a|^2)] + \tau[\Phi(|b|^2)] \right).$$

This concludes (3.3).

Now, to obtain the required inequality, it suffices to repeat the proof of the first part of Theorem 3.2 in [13] through using (3.3). We omit the details.  $\square$



**Remark 3.2.** For a sequences  $x = (x_n)$  in  $L_\Phi(\mathcal{M})$ , set

$$\left\| \sup_n^+ x_n \right\|_\Phi := \inf \left\{ \lambda > 0 : \tau \left[ \Phi \left( \sup_n^+ \frac{x_n}{\lambda} \right) \right] \leq 1 \right\}.$$

One can check that  $\|\sup_n^+ x_n\|_\Phi$  is a norm in  $x = (x_n)$ . Define

$$L_\Phi(\mathcal{M}; \ell^\infty) := \left\{ (x_n) \subset L_\Phi(\mathcal{M}) : \tau \left[ \Phi \left( \sup_n^+ \frac{x_n}{\lambda} \right) \right] < \infty \text{ for some } \lambda > 0 \right\},$$

equipped with  $\|(x_n)\|_{L_\Phi(\mathcal{M}; \ell^\infty)} = \|\sup_n^+ x_n\|_\Phi$ . Then  $L_\Phi(\mathcal{M}; \ell^\infty)$  is a Banach space. For  $1 \leq p < \infty$ , if  $\Phi(t) = t^p$  then  $L_\Phi(\mathcal{M}; \ell^\infty) = L_p(\mathcal{M}; \ell^\infty)$ , which was studied extensively in [21]. The details are left to the interested readers.

We are ready to state and prove the main result of this section.

**Theorem 3.2.** *Let  $S = (S_n)_{n \geq 0}$  be a sequence of maps from  $L_1^+(\mathcal{M}) + L_\infty^+(\mathcal{M}) \mapsto L_0^+(\mathcal{M})$ . Let  $1 \leq p < \infty$ . Assume that  $S$  is subadditive. If  $S$  is simultaneously of weak type  $(p, p)$  with constant  $C_p$  and of type  $(\infty, \infty)$  with constant  $C_\infty$ , then for an Orlicz function  $\Phi$  with  $p < p_\Phi \leq q_\Phi < \infty$ , there exists a positive constant  $C$  depending only on  $C_p, C_\infty, p_\Phi$  and  $q_\Phi$ , such that*

$$(3.4) \quad \tau \left[ \Phi \left( \sup_n^+ S_n(x) \right) \right] \leq C \tau [\Phi(x)],$$

for all  $x \in L_\Phi^+(\mathcal{M})$ .

*Proof.* Since  $S$  is of weak type  $(p, p)$  with constant  $C_p$ , for any  $x \in L_p^+(\mathcal{M})$  and each  $\lambda > 0$  there is a projection  $q^{(\lambda)} \in \mathcal{M}$  such that

$$\tau(1 - q^{(\lambda)}) \leq \frac{C_p^p \tau(|x|^p)}{\lambda^p} \quad \text{and} \quad q^{(\lambda)} S_n(x) q^{(\lambda)} \leq \lambda q^{(\lambda)}, \quad \forall n \geq 1.$$

For any  $k \in \mathbb{Z}$  we set

$$q_k = \bigwedge_{j \geq k} q^{(2^j)} \quad \text{and} \quad p_k = q_k - q_{k-1}.$$

We claim the following two facts.

(i)  $q_k S_n(x) q_k \leq 2^k q_k$  and

$$(3.5) \quad \tau(1 - q_k) \leq \frac{C_p^p}{1 - 2^{-p}} \frac{\tau(x^p)}{2^{kp}}, \quad \forall k \in \mathbb{Z}.$$

(ii) Suppose in addition, that  $x \in \mathcal{M}$ . Fix an integer  $N$  and a sequence  $(\alpha_k)_{k=-\infty}^N$  of positive numbers for which  $\sum_{k \leq N} \frac{2^k}{\alpha_k} < \infty$ . Then the operator

$$(3.6) \quad a = 2C_\infty \|x\| (1 - q_N) + 2 \left( \sum_{k \leq N} \frac{2^k}{\alpha_k} \right) \sum_{k \leq N} \alpha_k p_k.$$

is a majorant of  $S(x)$ , i.e.,  $S_n(x) \leq a$  for all  $n \geq 1$ .

To prove these two statements, note that

$$\tau(1 - q_k) \leq \sum_{j \geq k} \tau(1 - q^{(2^j)}) \leq C_p^p \tau(x^p) \sum_{j \geq k} 2^{-jp} = \frac{C_p^p}{1 - 2^{-p}} \frac{\tau(x^p)}{2^{kp}},$$

which proves (3.5). On the other hand, for a fixed  $\xi \in \mathbb{H}$  we have

$$\begin{aligned} (q_N S_n(x) q_N \xi, \xi) &= \left( \sum_{k, m \leq N} p_k S_n(x) p_m \xi, \xi \right) \\ &\leq \sum_{k, m \leq N} \|p_k S_n(x) p_m\| \|p_k \xi\| \|p_m \xi\| \\ &\leq \sum_{k, m \leq N} \|p_k S_n(x) p_k\|^{\frac{1}{2}} \|p_m S_n(x) p_m\|^{\frac{1}{2}} \|p_k \xi\| \|p_m \xi\| \\ &= \left( \sum_{k \leq N} \|p_k S_n(x) p_k\|^{\frac{1}{2}} \|p_k \xi\| \right)^2. \end{aligned}$$

Since  $p_k S_n(x) p_k \leq 2^k p_k$  and so  $\|p_k S_n(x) p_k\| \leq 2^k$ , one concludes that

$$(q_N S_n(x) q_N \xi, \xi) \leq \left( \sum_{k \leq N} \frac{2^k}{\alpha_k} \right) \sum_{k \leq N} \alpha_k \|p_k \xi\|^2 = (a_N \xi, \xi),$$

where  $a_N = \left( \sum_{k \leq N} \frac{2^k}{\alpha_k} \right) \sum_{k \leq N} \alpha_k p_k$ . Note that

$$S_n(x) \leq 2q_N S_n(x) q_N + 2(1 - q_N) S_n(x) (1 - q_N).$$

Thus,  $a$  is a majorant of  $S(x)$ .

Take  $x \in L_\Phi^+(\mathcal{M})$  and introduce

$$\tilde{x} = \sum_{i \in \mathbb{Z}} 2^{i+1} E_{(2^i, 2^{i+1}]}(x) = \sum_{i \in \mathbb{Z}} 2^i e_i,$$

where  $e_i = E_{(2^i, \infty)}(x)$ . For a fixed  $e_i$ , we will construct a suitable majorant of the sequence  $S(e_i) = (S_n(e_i))_{n \geq 1}$ . To this end, we take  $p < q < p_\Phi$  and set

$$\delta = \frac{1}{2} \left( \frac{1}{p} + \frac{1}{q} \right) \quad \text{and} \quad \alpha_k = h(2^{(N-k)p}),$$

where  $h(t) = \min\{t^{-\delta}, 1\}$  and  $N$  is the largest integer satisfying

$$\int_0^1 \frac{1}{h(t^{-p})} dt \leq \frac{C_\infty}{2^{N+1}}.$$

By (3.6) we obtain the corresponding majorant of the sequence  $S(e_i) = (S_n(e_i))_{n \geq 1}$ , denoted by  $a_i$ . We claim that there exists a constant  $C > 0$  depending only on  $C_p, C_\infty, p$  and  $q$  such that

$$(3.7) \quad \mu_t(a_i) \leq C h\left(\frac{t}{\tau(e_i)}\right) \quad \text{and} \quad \tilde{\mu}_t(a_i) \leq \frac{C}{1-\delta} h\left(\frac{t}{\tau(e_i)}\right), \quad \forall t > 0,$$

where  $\tilde{\mu}_t(x) = \frac{1}{t} \int_0^t \mu_s(x) ds$  for any  $x \in L_0(\mathcal{M})$  and all  $t > 0$ .

Indeed, an immediate computation yields that

$$\mu(a_i) \leq 2C_\infty \left( \chi_{[0, \tau(1-q_N))} + \sum_{k \leq N} 2^{-\delta(N-k)p} \chi_{(\tau(1-q_k), \tau(1-q_{k-1})]} \right).$$

By (3.5) one has

$$\mu(a_i) \leq 2C_\infty \left( \chi_{[0, C'_p 2^{-pN} \tau(e_i))} + \sum_{k \leq N} 2^{-\delta(N-k)p} \chi_{(C'_p 2^{-kp} \tau(e_i), C'_p 2^{-(k+1)p} \tau(e_i)]} \right),$$

where  $C'_p = C_p^p / (1 - 2^{-p})$ . Since for any  $t \in (C'_p 2^{-kp} \tau(e_i), C'_p 2^{-(k+1)p} \tau(e_i)]$ ,

$$h(2^{(N-k)p}) \leq h\left(2^{-p} \frac{2^{pN} t}{C'_p \tau(e_i)}\right),$$

we follow that

$$\begin{aligned}
\mu_t(a_i) &\leq 2C_\infty \left( \chi_{[0, C'_p 2^{-pN} \tau(e_i)]}(t) + h\left(2^{-p} \frac{2^{pN} t}{C'_p \tau(e_i)}\right) \chi_{(C'_p 2^{-pN} \tau(e_i), \infty)}(t) \right) \\
&= 2C_\infty \left[ \chi_{[0, 1]} \left( \frac{2^{pN} t}{C'_p \tau(e_i)} \right) + h\left(2^{-p} \frac{2^{pN} t}{C'_p \tau(e_i)}\right) \chi_{(1, \infty)} \left( \frac{2^{pN} t}{C'_p \tau(e_i)} \right) \right] \\
&= 2C_\infty h\left(2^{-p} \frac{2^{pN} t}{C'_p \tau(e_i)}\right) \\
&\leq Ch\left(\frac{t}{\tau(e_i)}\right).
\end{aligned}$$

This proves the first inequality in (3.7), from which the second one follows.

Since  $x \mapsto \tilde{\mu}(x)$  is sublinear, we have

$$\begin{aligned}
\tau \left[ \Phi \left( \sum_{i \in \mathbb{Z}} 2^i a_i \right) \right] &\leq \int_0^\infty \Phi \left[ \tilde{\mu}_t \left( \sum_{i \in \mathbb{Z}} 2^i a_i \right) \right] dt \\
&\leq \int_0^\infty \Phi \left( \sum_{i \in \mathbb{Z}} 2^i \tilde{\mu}_t(a_i) \right) dt \\
&\lesssim \int_0^\infty \Phi \left[ \sum_{i \in \mathbb{Z}} 2^i h\left(\frac{t}{\tau(e_i)}\right) \right] dt \\
&= \int_0^\infty \Phi \left[ \int_0^\infty \sum_{i \in \mathbb{Z}} 2^i \chi_{(0, \tau(e_i)]} \left( \frac{t}{s} \right) (-h'(s)) ds \right] dt.
\end{aligned}$$

Note that  $\mu_t(\tilde{x}) = \sum_{i \in \mathbb{Z}} 2^i \chi_{(0, \tau(e_i)]}(t)$  and hence we have

$$\begin{aligned}
\tau \left[ \Phi \left( \sum_{i \in \mathbb{Z}} 2^i a_i \right) \right] &\lesssim \int_0^\infty \Phi \left[ \int_0^\infty \mu_{\frac{t}{s}}(\tilde{x}) (-h'(s)) ds \right] dt \\
&\leq \int_0^\infty \Phi \left[ \int_0^\infty \tilde{\mu}_{\frac{t}{s}}(\tilde{x}) (-h'(s)) ds \right] dt.
\end{aligned}$$

Define  $T : L_1(\mathcal{M}) + L_\infty(\mathcal{M}) \mapsto L_1(0, \infty) + L_\infty(0, \infty)$  by

$$(Tx)(t) = \int_0^\infty \tilde{\mu}_{\frac{t}{s}}(x) (-h'(s)) ds, \quad \forall t > 0.$$

Then

$$\|Tx\|_q \leq \int_0^\infty \|\tilde{\mu}_{\frac{t}{s}}(x)\|_q (-h'(s)) ds = C_{p,q} \|\tilde{\mu}(x)\|_q \leq C_{p,q} \|x\|_{L_q(\mathcal{M})},$$

where the last inequality is obtained by the classical Hardy-Littlewood inequality: the mapping  $f \mapsto \frac{1}{t} \int_0^t |f(s)| ds$  is bounded in  $L_q(0, \infty)$  provided  $1 < q \leq \infty$ . Also, it is easy to check that  $T$  is of type  $(\infty, \infty)$ . Thus, by Theorem 2.1 in [4] we conclude that

$$\tau \left[ \Phi \left( \sum_{i \in \mathbb{Z}} 2^i a_i \right) \right] \lesssim \int_0^\infty \Phi \left[ \int_0^\infty \tilde{\mu}_{\frac{t}{s}}(\tilde{x}) (-h'(s)) ds \right] dt \lesssim \tau[\Phi(\tilde{x})].$$

Since  $\tilde{x} \leq 2x$  and

$$S_n(x) \leq S_n(\tilde{x}) \leq \sum_{i \in \mathbb{Z}} 2^i a_i, \quad \forall n \geq 1,$$

we conclude (3.4).  $\square$

**Remark 3.3.** (1) The classical Marcinkiewicz interpolation theorem has been extended to include Orlicz spaces as interpolation classes by A. Zygmund, A. P. Calderón et al. (for references see [26]). The noncommutative analogue of this was recently obtained in [4]. Theorem 3.2 can be considered as a noncommutative analogue of the Marcinkiewicz type interpolation theorem for convex functions of maximal functions.

- (2) One could expect that Theorem 3.2 should be valid under the assumption that  $S$  is simultaneously of weak type  $(p_0, p_0)$  and of type  $(p_1, p_1)$  and  $\Phi$  an Orlicz function with  $1 \leq p_0 < p_\Phi \leq q_\Phi < p_1 \leq \infty$ . At the time of this writing, this question remains open. However, this is indeed the case for weak type moments of maximal functions, see Theorem 5.1 below.

#### 4. PROOFS OF MAIN RESULTS

Let  $\Phi$  be an Orlicz function. As noted in [4] (Remark 1.1 there), if  $1 < p_\Phi \leq q_\Phi < \infty$ , then for any noncommutative  $L_\Phi$ -martingale  $x = (x_n)$ , there exists a unique  $x_\infty \in L_\Phi(\mathcal{M})$  such that  $x_n = \mathcal{E}_n(x_\infty)$  for all  $n$ . We simply write  $x_\infty = x$  in this case.

First of all, we prove the noncommutative analogue of the Doob inequality for convex functions of maximal functions on martingales.

*Proof of Theorem 1.2.* Decomposing an operator into a linear combination of four positive ones, we can assume that  $x = (x_n)$  is a positive martingale in  $L_\Phi(\mathcal{M})$ . Let  $S = (\mathcal{E}_n)$ . By Cuculescu's weak type  $(1, 1)$  maximal martingale inequality [11], we see that  $S$  is of weak type  $(1, 1)$ . Also, by Junge's noncommutative Doob inequality in  $p = \infty$  we have that  $S$  is of type  $(\infty, \infty)$ . Thus, by Theorem 3.2 we conclude that

$$\tau \left[ \Phi \left( \sup_n^+ x_n \right) \right] \lesssim \tau [\Phi(|x|)].$$

To prove the converse inequality, consider a decomposition  $x_n = ay_nb$  for all  $n$  and  $\sup_n \|y_n\|_\infty \leq 1$ . One has

$$\begin{aligned} \tau [\Phi(|x|)] &\leq \int_0^\infty \Phi(\mu_t(x)) dt \leq \sup_n \|y_n\|_\infty \int_0^\infty \Phi(\mu_t(|a||b|)) dt \\ &\leq 2 \sup_n \|y_n\|_\infty \int_0^\infty \Phi[\mu_t(|a|)\mu_t(|b|)] dt \\ &\leq 2 \sup_n \|y_n\|_\infty \int_0^\infty \Phi \left[ \frac{1}{2} (\mu_t(|a|)^2 + \mu_t(|b|)^2) \right] dt \\ &\leq 2 \sup_n \|y_n\|_\infty \frac{1}{2} (\tau [\Phi(|a|^2)] + \tau [\Phi(|b|^2)]). \end{aligned}$$

Thus,

$$\tau [\Phi(|x|)] \leq 2\tau \left[ \Phi \left( \sup_n^+ x_n \right) \right].$$

This completes the proof.  $\square$

**Remark 4.1.** Let  $\Phi$  be an Orlicz function. We define the Orlicz maximal space of noncommutative martingales as

$$L_\Phi^{\max}(\mathcal{M}) := \{x \in L_\Phi(\mathcal{M}) : \|x\|_{L_\Phi^{\max}} = \|\sup_n^+ \mathcal{E}_n(x)\|_\Phi < \infty\}.$$

Then, Theorem 1.2 implies that  $L_\Phi^{\max}(\mathcal{M}) = L_\Phi(\mathcal{M})$  with equivalent norms provided  $1 < p_\Phi \leq q_\Phi < \infty$ .

Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* It is proved in [4] that if  $1 < p_\Phi \leq q_\Phi < 2$ , then

$$(4.1) \quad \tau [\Phi(|x|)] \approx \inf \left\{ \tau \left( \Phi \left[ \left( \sum_{k=0}^\infty |dy_k|^2 \right)^{\frac{1}{2}} \right] \right) + \tau \left( \Phi \left[ \left( \sum_{k=0}^\infty |dz_k^*|^2 \right)^{\frac{1}{2}} \right] \right) \right\},$$

where the infimum runs over all decomposition  $x_n = y_n + z_n$  with  $y_n$  in  $\mathcal{H}_\Phi^C(\mathcal{M})$  and  $z_n$  in  $\mathcal{H}_\Phi^R(\mathcal{M})$ ; and if  $2 < p_\Phi \leq q_\Phi < \infty$ , then

$$(4.2) \quad \tau[\Phi(|x|)] \approx \max \left\{ \tau \left( \Phi \left[ \left( \sum_{k=0}^{\infty} |dx_k|^2 \right)^{\frac{1}{2}} \right] \right), \tau \left( \Phi \left[ \left( \sum_{k=0}^{\infty} |dx_k^*|^2 \right)^{\frac{1}{2}} \right] \right) \right\}.$$

An appeal to (1.4) yields the required inequalities (1.2) and (1.3).  $\square$

**Remark 4.2.** We note that there is a gap in the proof of (4.2) in [4], as pointed out to us by Q.Xu. This was recently resolved in [14].

Then we turn to the proof of Theorem 1.3.

*Proof of Theorem 1.3.* Decomposing an operator into a linear combination of four positive ones, we can assume  $x \in L_\Phi^+(\mathcal{M})$ . Let  $S = (M_n)$ . Each  $M_n$  is considered to be a map on  $L_1^+(\mathcal{M}) + L_\infty^+(\mathcal{M})$ , positive and additive (and so subadditive too). Yeadon's weak type (1, 1) maximal ergodic inequality in [37] says that  $S$  is of weak type (1, 1). Also,  $S$  is evidently of type  $(\infty, \infty)$ . Then, Theorem 3.2 concludes (1.5).

On the other hand, let  $S = (T^n)$ . Then  $S$  is additive and so subadditive. By Theorem 5.1 in [21],  $S$  is of type  $(p, p)$  for every  $1 < p \leq \infty$ . An appeal to Theorem 3.2 immediately yields (1.6).  $\square$

Let us present two examples illustrating applications of the inequalities obtained above.

**Example 4.1.** Let  $\Phi(t) = t^a \ln(1 + t^b)$  with  $a > 1$  and  $b > 0$ . It is easy to check that  $\Phi$  is an Orlicz function and

$$p_\Phi = a \quad \text{and} \quad q_\Phi = a + b.$$

Thus, both Theorems 1.2 and 1.3 can be applied to this function. Furthermore, if  $1 < a < a + b < 2$ , then (1.2) holds true; if  $a > 2$ , then (1.3) is valid. Unfortunately, when  $1 < a \leq 2 \leq a + b$ , then Theorem 1.1 gives no information.

**Example 4.2.** Let  $\Phi(t) = t^p(1 + c \sin(p \ln t))$  with  $p > 1/(1 - 2c)$  and  $0 < c < 1/2$ . Then  $\Phi$  is an Orlicz function and

$$p_\Phi = q_\Phi = p.$$

Therefore, Theorems 1.2 and 1.3 can be applied to this function, and so does Theorem 1.1 except the case  $p = 2$ .

## 5. WEAK TYPE INEQUALITIES

All the results continue to hold if we replace the noncommutative maximal function  $\tau[\Phi(\sup_n^+ x_n)]$  by a certain weak maximal function, as considered in [5]. The required modifications are not difficult and left to the interested reader. However, for the sake of convenience, we write the corresponding definitions and results, and some main points of Theorem 5.1. We refer to [5] for noncommutative weak Orlicz spaces and for the terminology used here.

Let  $\Phi$  be an Orlicz function. For  $x \in L_\Phi^w(\mathcal{M})$ , we set

$$\|x\|_{\Phi, \infty} = \sup_{t > 0} t \Phi[\mu_t(x)].$$

When  $\Phi(t) = t^p$ ,  $\|x\|_{\Phi, \infty}$  is just the usual weak  $L_p$ -norm  $\|x\|_{p, \infty}$ .

**Definition 5.1.** Let  $(x_n)$  be a sequence in  $L_\Phi^w(\mathcal{M})$ . We define  $\|\sup_n^+ x_n\|_{\Phi, \infty}$  by

$$(5.1) \quad \|\sup_n^+ x_n\|_{\Phi, \infty} := \inf \left\{ \frac{1}{2} \left( \| |a|^2 \|_{\Phi, \infty} + \| |b|^2 \|_{\Phi, \infty} \right) \sup_n \|y_n\|_\infty \right\}$$

where the infimum is taken over all decompositions  $x_n = ay_nb$  for  $a, b \in L_0(\mathcal{M})$  and  $(y_n) \subset L_\infty(\mathcal{M})$  with  $|a|^2, |b|^2 \in L_\Phi^w(\mathcal{M})$ , and  $\|y_n\|_\infty \leq 1$  for all  $n$ .

**Theorem 5.1.** *Suppose  $1 \leq p_0 < p_1 \leq \infty$ . Let  $S = (S_n)_{n \geq 0}$  be a sequence of maps from  $L_{p_0}^+(\mathcal{M}) + L_{p_1}^+(\mathcal{M}) \mapsto L_0^+(\mathcal{M})$ . Assume that  $S$  is subadditive. If  $S$  is of weak type  $(p_0, p_0)$  with constant  $C_0$  and of type  $(p_1, p_1)$  with constant  $C_1$ , then for an Orlicz function  $\Phi$  with  $p_0 < a_\Phi \leq b_\Phi < p_1$ , there exists a positive constant  $C$  depending only on  $p_0, p_1, C_0, C_1$  and  $\Phi$ , such that*

$$(5.2) \quad \left\| \sup_n^+ S_n(x) \right\|_{\Phi, \infty} \leq C \|x\|_{\Phi, \infty},$$

for all  $x \in L_\Phi^w(\mathcal{M})_+$ .

*Proof.* We give the main point of the proof. Indeed, modifying slightly the proof of Theorem 3.1 in [21] we conclude that for  $p_0 < p'_0 < a_\Phi \leq b_\Phi < p'_1 < p_1 \leq \infty$ ,

$$\left\| \sup_n^+ S_n(x) \right\|_{p'_i, \infty} \leq C_{p'_i} \|x\|_{p'_i, \infty}, \quad i = 0, 1,$$

that is, for each  $x_i \in L_{p'_i}^+(\mathcal{M})$  there exists  $a_i \in L_{p'_i}^+(\mathcal{M})$  such that

$$(5.3) \quad \|a_i\|_{p'_i, \infty} \leq C \|x_i\|_{p'_i, \infty} \quad \text{and} \quad S_n(x_i) \leq a_i, \quad \forall n \geq 1.$$

(This can be also obtained by Theorem 5.4 below.)

Now, take  $x \in L_\Phi^w(\mathcal{M})_+$ . For any  $\alpha > 0$  let  $x = x_0^\alpha + x_1^\alpha$ , where  $x_0^\alpha = x e_{(\alpha, \infty)}(x)$ . By (5.3), for  $x_i^\alpha$  there exists a corresponding  $a_i$  ( $i = 0, 1$ ). The remainder of the proof is the same as that of the proof of Theorem 4.2 in [5].  $\square$

The following is the weak type Doob inequality for noncommutative martingales.

**Theorem 5.2.** *Let  $\mathcal{M}$  be a finite von Neumann algebra with a normalized normal faithful trace  $\tau$ , equipped with a filtration  $(\mathcal{M}_n)$  of von Neumann subalgebras of  $\mathcal{M}$ . Let  $\Phi$  be an Orlicz function and let  $x = (x_n)$  be a noncommutative  $L_\Phi^w$ -martingale with respect to  $(\mathcal{M}_n)_{n \geq 0}$ . If  $1 < a_\Phi \leq b_\Phi < \infty$ , then*

$$(5.4) \quad \left\| \sup_n^+ x_n \right\|_{\Phi, \infty} \approx \|x\|_{\Phi, \infty}.$$

Combining this with Theorem 5.8 in [5] and the corresponding result in [14] we have the following weak type moment Burkholder-Davis-Gundy inequality for noncommutative martingales.

**Theorem 5.3.** *Let  $\mathcal{M}$  be a finite von Neumann algebra with a normalized normal faithful trace  $\tau$ , equipped with a filtration  $(\mathcal{M}_n)$  of von Neumann subalgebras of  $\mathcal{M}$ . Let  $\Phi$  be an Orlicz function and let  $x = (x_n)_{n \geq 0}$  be a noncommutative  $L_\Phi$ -martingale with respect to  $(\mathcal{M}_n)_{n \geq 0}$ . If  $1 < a_\Phi \leq b_\Phi < 2$ , then*

$$(5.5) \quad \left\| \sup_n^+ x_n \right\|_{\Phi, \infty} \approx \inf \left\{ \left\| \left( \sum_{k=0}^{\infty} |dy_k|^2 \right)^{\frac{1}{2}} \right\|_{\Phi, \infty} + \left\| \left( \sum_{k=0}^{\infty} |dz_k^*|^2 \right)^{\frac{1}{2}} \right\|_{\Phi, \infty} \right\},$$

where the infimum runs over all decomposition  $x_n = y_n + z_n$  with  $(y_n)$  in  $L_\Phi^w(\mathcal{M}; \ell_C^2)$  and  $(z_n)$  in  $L_\Phi^w(\mathcal{M}; \ell_R^2)$ ; and if  $2 < a_\Phi \leq b_\Phi < \infty$ , then

$$(5.6) \quad \left\| \sup_n^+ x_n \right\|_{\Phi, \infty} \approx \left\| \left( \sum_{k=0}^{\infty} |dx_k|^2 \right)^{\frac{1}{2}} \right\|_{\Phi, \infty} + \left\| \left( \sum_{k=0}^{\infty} |dx_k^*|^2 \right)^{\frac{1}{2}} \right\|_{\Phi, \infty}.$$

The weak type moment analogue of Theorem 1.3 concerning maximal ergodic inequalities is similar and omitted.

Moreover, we can obtain the associated maximal inequalities on noncommutative symmetric spaces. Let  $E$  be a rearrangement invariant (r.i., in short) Banach space and  $E(\mathcal{M}, \tau)$  the associated noncommutative symmetric space. For details on r.i. spaces we

refer to [24] and for the theory of noncommutative symmetric spaces to [15, 16, 23, 35]. Then we have

**Theorem 5.4.** *Let  $S = (S_n)_{n \geq 0}$  be a sequence of maps from  $L_1^+(\mathcal{M}) + L_\infty^+(\mathcal{M}) \mapsto L_0^+(\mathcal{M})$ . Assume that  $S$  is subadditive. Let  $1 \leq p < \infty$ . Let  $E$  be a rearrangement invariant space with the upper Boyd index  $p_E > p$ . If  $S$  is simultaneously of weak type  $(p, p)$  with constant  $C_p$  and of type  $(\infty, \infty)$  with constant  $C_\infty$ , then there exists a positive constant  $C_E$  depending only on  $C_p, C_\infty, p$  and  $p_E$ , such that for any  $x \in E^+(\mathcal{M}, \tau)$  there exists  $a \in E^+(\mathcal{M}, \tau)$  satisfying*

$$(5.7) \quad \|a\|_{E(\mathcal{M}, \tau)} \leq C_E \|x\|_{E(\mathcal{M}, \tau)} \quad \text{and} \quad S_n(x) \leq a, \quad \forall n \geq 0.$$

*Proof.* Indeed, the construction of the majorant of  $S = (S_n(\tilde{x}))$  in the proof for Theorem 3.2 is clearly valid. Hence, we have

$$\begin{aligned} \left\| \sum_{i \in \mathbb{Z}} 2^i a_i \right\|_{E(\mathcal{M}, \tau)} &\lesssim \left\| \int_0^\infty \mu_{\tilde{s}}(\tilde{x})(-h'(s)) ds \right\|_E \\ &\leq \int_0^\infty \|D_s \mu(\tilde{x})\|_E (-h'(s)) ds \\ &\leq \int_0^\infty \|D_s\|_E (-h'(s)) ds \|\tilde{x}\|_{E(\mathcal{M}, \tau)}. \end{aligned}$$

Here  $D_s$  ( $0 < s < \infty$ ) are linear operators acting on measurable functions  $f$  on  $(0, \infty)$  defined by

$$(D_s f)(t) = f(t/s), \quad 0 < t < \infty.$$

It is known that for  $1 < q < p_E$  there is a constant  $C_{E,q} > 0$  such that

$$\|D_s\|_E \leq C_{E,q} s^{\frac{1}{q}}, \quad \forall 1 < s < \infty.$$

Thus

$$\int_0^\infty \|D_s\|_E (-h'(s)) ds \leq C_{E,q} \int_1^\infty s^{\frac{1}{q}-\delta-1} ds = C_{p_E,q,p} < \infty.$$

This completes the proof.  $\square$

Now, the expected results on noncommutative symmetric spaces are in order, including the Doob maximal martingale inequality, Dunford-Schwartz and Stein maximal ergodic inequalities, as well as the corresponding pointwise convergence theorems (see [21] for detailed information). We omit the details.

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